

THE GROMOV HYPERBOLICITY OF 5/9-COMPLEXES

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ABSTRACT. We introduce and study a local combinatorial condition, called the 5/9-condition, on a simplicial complex, implying Gromov hyperbolicity of its universal cover. We hereby give an application of another combinatorial condition, called 8-location, introduced by Osajda. Along the way we prove the minimal filling diagram lemma for 5/9-complexes.

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1. INTRODUCTION

Curvature can be expressed both in metric and combinatorial terms. Metrically, one can refer to nonpositively curved (respectively, negatively curved) metric spaces in the sense of Aleksandrov, i.e. by comparing small triangles in the space with triangles in the Euclidean plane (hyperbolic plane). These are the CAT(0) (respectively, CAT(-1)) spaces. Combinatorially, one looks for local combinatorial conditions implying some global features typical for nonpositively curved metric spaces.

A very important combinatorial condition of this type was formulated by Gromov [Gro87] for cubical complexes, i.e. cellular complexes with cells being cubes. Namely, simply connected cubical complexes with links (that can be thought as small spheres around vertices) being flag (respectively, 5-large, i.e. flag-no-square) simplicial complexes carry a canonical CAT(0) (respectively, CAT(-1)) metric. Another important local combinatorial condition is local k -largeness, introduced by Januszkiewicz-Świątkowski [JŚ06] and Haglund [Hag03]. A flag simplicial complex is *locally k -large* if its links do not contain 'essential' loops of length less than k . In particular, simply connected locally 7-large simplicial complexes, i.e. 7-systolic complexes, are Gromov hyperbolic [JŚ06]. The theory of 7-systolic groups, that is groups acting geometrically on 7-systolic complexes, allowed to provide important examples of highly dimensional Gromov hyperbolic groups [JŚ03, JŚ06, Osa13, OŚ15].

However, for groups acting geometrically on CAT(-1) cubical complexes or on 7-systolic complexes, some very restrictive limitations are known. For example, 7-systolic groups are in a sense 'asymptotically hereditarily aspherical', i.e. asymptotically they can not contain essential spheres. This yields in particular that such groups are not fundamental groups of negatively curved manifolds of dimension above two; see e.g. [JŚ07, Osa07, Osa08, OŚ15, GO14, Osa15b]. This rises need for other combinatorial conditions, not imposing restrictions as above. In

[Osa13b, CO15, BCC⁺13, CCHO14] some conditions of this type are studied – they form a way of unifying CAT(0) cubical and systolic theories. On the other hand, Osajda [Osa15] introduced a local combinatorial condition of 8-*location*, and used it to provide a new solution to Thurston’s problem about hyperbolicity of some 3-manifolds.

In [Laz15] a systematic study of a version of 8-*location*, suggested in [Osa15, Subsection 5.1], is undertaken. This version is in a sense more natural than the original one (tailored to Thurston’s problem), and none of them is implied by the other. However, in the new 8-*location* essential 4-loops are allowed. This suggests that it can be used in a much wider context. Roughly (see Section 2 for the precise definition), the new 8-*location* says that essential loops of length at most 8 admit filling diagrams with at most one internal vertex. In [Laz15] (Theorem 4.3) it is shown that this local combinatorial condition is a negative-curvature-type condition, by proving that simply connected, 8-located simplicial complexes are Gromov hyperbolic. In the current paper we give an application to this result. Namely, we introduce another combinatorial curvature condition, called the 5/9-condition, and we show that the complexes which satisfy it, are also Gromov hyperbolic. Our proof relies on the minimal filling diagrams lemma for 5/9-complexes. We prove that the disc in the diagram associated to a cycle in a simply connected 5/9-complex is itself a 5/9-complex (Lemma 3.1). Further we show that a disc satisfying the 5/9-condition, is 8-located (Lemma 4.1). Therefore the 5/9-complex itself is 8-located and hence, by [Laz15], Theorem 4.3, Gromov hyperbolic (Theorem 4.2). Our proof for the minimal filling diagram lemma is similar to the one given for the systolic case in [JŠ06] and [Pry14].

The paper’s main result is already mentioned without proof in [Osa15] (Section 5.3) as an application concerning some weakly systolic complexes and groups. Because the 5/9-condition implies the hyperbolicity of weakly systolic complexes, it is more general than the well studied conditions for hyperbolicity: local 7-largeness for 7-systolic complexes (see [JŠ06], Section 2), and $SD_2^*(7)$ for weakly systolic complexes (see [Osa13b], Section 7).

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2. PRELIMINARIES

Let X be a simplicial complex. We denote by $X^{(k)}$ the k -skeleton of X , $0 \leq k < \dim X$. A subcomplex L in X is called *full* (in X) if any simplex of X spanned by a set of vertices in L , is a simplex of L . For a set $A = \{v_1, \dots, v_k\}$ of vertices of X , by $\langle A \rangle$ or by $\langle v_1, \dots, v_k \rangle$ we denote the *span* of A , i.e. the smallest full subcomplex of X that contains A . We write $v \sim v'$ if $\langle v, v' \rangle \in X$ (it can happen that $v = v'$). We write $v \nsim v'$ if $\langle v, v' \rangle \notin X$. We call X *flag* if any finite set of vertices, which are pairwise connected by edges of X , spans a simplex of X .

A *cycle* (loop) γ in X is a subcomplex of X isomorphic to a triangulation of S^1 . A k -*wheel* in X ($v_0; v_1, \dots, v_k$) (where $v_i, i \in \{0, \dots, k\}$ are vertices of X) is a subcomplex of X such that (v_1, \dots, v_k) is a full cycle and $v_0 \sim v_1, \dots, v_k$. The *length* of γ (denoted by $|\gamma|$) is the number of edges in γ . If $v_i \sim v_{i+j}$, $1 \leq i \leq k$, $2 \leq j \leq k-i$ then we call $\langle v_i, v_{i+j} \rangle$ a *diagonal* of the cycle γ .

We define the *metric* on the 0-skeleton of X as the number of edges in the shortest 1-skeleton path joining two given vertices and we denote it by d . We call a vertex v a *neighbor* of another vertex w if $d(v, w) = 1$. A *ball* (*sphere*) $B_i(v, X)$ ($S_i(v, X)$) of radius i around some vertex v is a full subcomplex of X spanned by vertices at distance at most i (at distance i) from v .

Definition 2.1. A simplicial complex is *m-located* if it is flag and every full homotopically trivial loop of length at most m is contained in a 1-ball.

The *link* of X at σ , denoted X_σ , is the subcomplex of X consisting of all simplices of X which are disjoint from σ and which, together with σ , span a simplex of X . A *full cycle* in X is a cycle that is full as subcomplex of X . We call a flag simplicial complex *k-large* if there are no full j -cycles in X , for $j < k$. We say X is *locally k-large* if all its links are *k-large*. We say a vertex of X is *k-large* if its link is *k-large*.

Definition 2.2. We say that a flag simplicial complex satisfies the *5/9-condition*, or that it is a *5/9-complex*, if it satisfies the following three conditions:

- (5/9): every vertex adjacent to a non-5-large (but 4-large) vertex is 9-large;
- (6/8): every vertex adjacent to a non-6-large (but 5-large) vertex is 8-large;
- (7/7): every vertex adjacent to a non-7-large (but 6-large) vertex is 7-large.

Definition 2.3. Let γ be a cycle in X . A *filling diagram* for γ is a simplicial map $f : D \rightarrow X$, where D is a triangulated 2-disc, and $f|_{\partial D}$ maps ∂D isomorphically onto γ . We denote a filling diagram for γ by (f, D) and we say it is:

- *minimal* if D consists of the least possible number of 2-simplices among filling diagrams for γ ;
- *nondegenerate* if f is a nondegenerate map;
- *locally k-large* if D is a locally *k-large* simplicial complex.

The disc D in the definition above is not necessarily simplicial, i.e. it may have multiple edges and loops. But, as required in the definition, the diagram is locally *k-large* only if the disc D is simplicial. A simplicial map with a nonsimplicial disc is defined as in the simplicial case.

Lemma 2.1. *Let X be a k -large simplicial complex, $m < k$ and let S_m^1 denote the subdivision of the circle into m edges. Then for any simplicial map $f_0 : S_m^1 \rightarrow X$ there exists a simplicial map $f : D \rightarrow X$, where D is a triangulated 2-disc, such that $\partial D = S_m^1$, $f|_{\partial D} = f_0$ and D has no interior vertices.*

Proof. For the proof see [JS06] (Lemma 1.3), and [Pry14] (Lemma 2.4). □

3. MINIMAL FILLING DIAGRAMS

In this section we show that the minimal diagram lemma holds for the 5/9-condition. A similar result for the systolic case is given in [JS06], Lemma 1.6 and Lemma 1.7, and in [Pry14], Theorem 2.7.

Lemma 3.1 (5/9-diagrams). *Let X be a 5/9-complex and let γ be a homotopically trivial loop in X . Then:*

- (1) *there exists a filling diagram for γ ;*
- (2) *any minimal filling diagram for γ is simplicial, nondegenerate and it satisfies the 5/9-condition.*

Proof. (1) For the proof see [JŠ06], Lemma 1.6 and [Pry14] (Theorem 2.7).
The proof uses Lemma 2.1

- (2) Let (D, f) be a minimal filling diagram for γ . Then D is simplicial and nondegenerate. For the proof see [JŠ06] (Lemma 1.6 and Lemma 1.7) and [Pry14] (Theorem 2.7).

We show further that D is a 5/9-complex. We must verify three conditions:

- (5/9): every vertex adjacent to a non-5-large (but 4-large) vertex is 9-large;
- (6/8): every vertex adjacent to a non-6-large (but 5-large) vertex is 8-large;
- (7/7): every vertex adjacent to a non-7-large (but 6-large) vertex is 7-large.

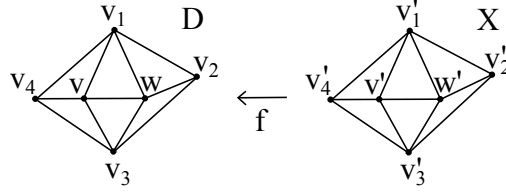
We will show only the first condition in 5 steps. The other two conditions can be shown similarly.

- (a) Step 1: (5/5): every vertex adjacent to a non-5-large (but 4-large) vertex is 5-large.

Assume D is such that $\partial D = \gamma = (v_1, v_2, v_3, v_4)$ is a cycle formed by two edges $(\langle v_1, v_4 \rangle, \langle v_3, v_4 \rangle)$ in the link of a vertex v and two edges in the link of a vertex w $(\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle)$ as shown in the figure below. v and w are interior vertices of D such that $v \sim w$. The link of v is 4-large but not 5-large and the link of its neighbor w is also 4-large but not 5-large. Let $\beta = (v_1, v_2, v_3, v)$ be the 4-cycle in the link of w . Let $\alpha = (v_1, w, v_3, v_4)$ be the 4-cycle in the link of v . Let γ' be a cycle in X such that (D, f) is a minimal filling diagram for γ' . Since the map f is nondegenerate and simplicial, there exist two vertices v', w' in X such that $f(v) = v'$ and $f(w) = w'$. Moreover, both v' and w' have a 4-cycle in their link whereas two of the edges of these 4-cycles coincide (see the figure below). Let $\beta' = (v'_1, v'_2, v'_3, v')$ be the 4-cycle in the link of w' . Let $\alpha' = (v'_1, w', v'_3, v'_4)$ be the 4-cycle in the link of v' . Because X satisfies the 5/9-condition, since v' is 4-large and not 5-large, every its neighbor should be 9-large. Thus, since the 4-cycle in the link of w' is not allowed to be full, it has a diagonal. Assume that $\langle v', v'_2 \rangle$ is a diagonal of the cycle β' . Because the map f is simplicial and nondegenerate, in D we have $v \sim v_2$. Note that therefore D is not flag. But D can be made flag because by Lemma 2.1, it can be triangulated with fewer triangles: $\langle v_1, v_2, v \rangle, \langle v_2, v_3, v \rangle, \langle v_1, v_4, v \rangle, \langle v_3, v_4, v \rangle$. The original triangulation of D had 6 triangles: $\langle v_1, v_2, w \rangle, \langle v_2, v_3, w \rangle, \langle v_3, v, w \rangle, \langle v_1, v, w \rangle, \langle v_3, v_4, v \rangle, \langle v_1, v_4, v \rangle$. Because the filling diagram (D, f) was chosen such that it is minimal, we have reached a contradiction. So the link of every neighbor of a vertex of D with 4-large but not 5-large link, is 5-large.

- (b) Step 2: (5/6): every vertex adjacent to a non-5-large (but 4-large) vertex is 6-large.

Assume D is such that $\partial D = \gamma = (v_1, v_2, v_3, v_4, v_5)$ is a cycle formed by two edges $(\langle v_1, v_5 \rangle, \langle v_4, v_5 \rangle)$ in the link of a vertex v and three edges $(\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle, \langle v_3, v_4 \rangle)$ in the link of a vertex w as shown in the figure below. v and w are interior vertices of D such that $v \sim w$. The



Step 1

link of v is 4-large but not 5-large and its neighbors w has a 5-large but not 6-large link. Let $\beta = (v_1, v_2, v_3, v_4, v)$ be the 5-cycle in the link of w . Let $\alpha = (v_1, w, v_4, v_5)$ be the 4-cycle in the link of v . Let γ' be a cycle in X such that (D, f) is a minimal filling diagram for γ' . Since the map f is simplicial and nondegenerate, there exist two vertices v', w' in X such that $f(v) = v'$ and $f(w) = w'$. Moreover, v' has a 4-cycle $\alpha' = (v'_1, w', v'_4, v'_5)$ in its link and w' has a 5-cycle $\beta' = (v'_1, v'_2, v'_3, v'_4, v')$ in its link whereas two of the edges of these cycles coincide (see the figure below). Because X satisfies the 5/9-condition, since v' is 4-large and not 5-large, every its neighbor should be 9-large. Hence, since the 5-cycle in the link of w' is not allowed to be full, it has a diagonal.

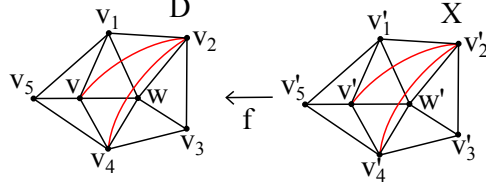
If the diagonal of the cycle β' is $\langle v'_1, v'_4 \rangle$, because the map f is simplicial and nondegenerate, in D we have $v_1 \sim v_4$. Note that then D is not flag. But, according to Lemma 2.1, we can triangulate D with 5 triangles: $\langle v_1, v_2, w \rangle$, $\langle v_2, v_3, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_1, v_4, w \rangle$, $\langle v_1, v_4, v_5 \rangle$. The diagram (D, f) was chosen such that it is minimal. Since it had 7 triangles ($\langle v_1, v_2, w \rangle$, $\langle v_2, v_3, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_4, v, w \rangle$, $\langle v_1, v, w \rangle$, $\langle v_1, v_5, w \rangle$, $\langle v_4, v_5, w \rangle$), we have reached a contradiction.

Consider further the case when $v'_1 \sim v'_4$. If the diagonal of the cycle β' is $\langle v', v'_2 \rangle$, because f is nondegenerate and simplicial, in D we have $v \sim v_2$. So we can triangulate D with 7 triangles: $\langle v_1, v_2, v \rangle$, $\langle v_2, v, w \rangle$, $\langle v_2, v_3, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_4, v, w \rangle$, $\langle v_1, v_5, v \rangle$, $\langle v_4, v_5, v \rangle$. Since the original triangulation of D which was minimal also had 7 triangles ($\langle v_1, v_2, w \rangle$, $\langle v_2, v_3, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_4, v, w \rangle$, $\langle v_1, v, w \rangle$, $\langle v_1, v_5, v \rangle$, $\langle v_4, v_5, v \rangle$), the new triangulation may also be considered. But (v_2, v_3, v_4, v) is a 4-cycle in the link of w in D . Such 4-cycle is, according to Step 1, forbidden. So we have reached a contradiction.

If the diagonal of the cycle β' is $\langle v'_2, v'_4 \rangle$, note that there is a 4-cycle (v'_1, v'_2, v'_4, v') in the link of w' in X . Since X is a 5/9-complex, any cycle in the link of w' should be of length 9 or more. So (v'_1, v'_2, v'_4, v') is not allowed to be full and hence it must have a diagonal. If $v' \sim v'_2$, since f is simplicial and nondegenerate, in D we have $v \sim v_2$ and $v_2 \sim v_4$. According to Lemma 2.1 we can change the triangulation of D by considering the triangles $\langle v_1, v_2, v \rangle$, $\langle v_2, v_4, v \rangle$, $\langle v_2, v_4, w \rangle$, $\langle v_2, v_3, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_1, v_5, v \rangle$, $\langle v_4, v_5, v \rangle$. Note that the new triangulation has 7 triangles like the original one which was minimal. So the new triangulation may also be considered. But (v_1, v_2, v_4, v) is a

4-cycle in the link of w in D . According to the previous step, such cycle is forbidden. This implies a contradiction.

So the link of every neighbor of a vertex of D whose link is 4-large but not 5-large, is 6-large.



Step 2

- (c) Step 3: (5/7): every vertex adjacent to a non-5-large (but 4-large) vertex is 7-large.

Assume D is such that $\partial D = \gamma = (v_1, v_2, v_3, v_4, v_5, v_6)$ is a cycle formed by two edges $(\langle v_1, v_6 \rangle, \langle v_5, v_6 \rangle)$ in the link of a vertex v and four edges $(\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle, \langle v_3, v_4 \rangle, \langle v_4, v_5 \rangle)$ in the link of a vertex w . The link of v is 4-large but not 5-large and its neighbor w has a 6-large but not 7-large link. Let $\beta = (v_1, v_2, v_3, v_4, v_5, v)$ be the 6-cycle in the link of w . Let $\alpha = (v_1, w, v_5, v_6)$ be the 4-cycle in the link of v . Let γ' be a cycle in X such that (D, f) is a minimal filling diagram for γ' . Since the map f is nondegenerate and simplicial, there exist two vertices v', w' in X such that $f(v) = v'$ and $f(w) = w'$. Moreover, v' has a 4-cycle $\alpha' = (v'_1, w', v'_5, v'_6)$ in its link and w' has a 6-cycle $\beta' = (v'_1, v'_2, v'_3, v'_4, v'_5, v')$ in its link whereas two of the edges of these cycles coincide. Because X satisfies the 5/9-condition, since v' is 4-large and not 5-large, every its neighbor should be 9-large. Thus, since the 6-cycle in the link of w' is not allowed to be full, it must have a diagonal.

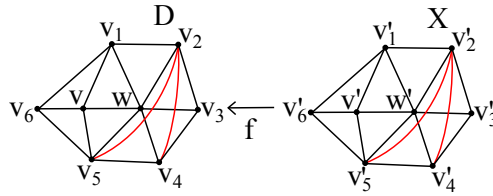
If the diagonal of the cycle β' is $\langle v'_1, v'_5 \rangle$, because the map f is simplicial and nondegenerate, in D we have $v_1 \sim v_5$. Note that then D is not flag. But D can be made flag since, according to Lemma 2.1, we can triangulate it with the triangles: $\langle v_1, v_2, w \rangle, \langle v_2, v_3, w \rangle, \langle v_3, v_4, w \rangle, \langle v_4, v_5, w \rangle, \langle v_1, v_5, w \rangle, \langle v_1, v_5, v_6 \rangle$. Since the new triangulation has fewer triangles than the original one which was minimal, we have reached a contradiction.

If the diagonal of the cycle β' is $\langle v'_1, v'_3 \rangle$, because the map f is simplicial and nondegenerate, in D we have $v_1 \sim v_3$. So according to Lemma 2.1, we can triangulate D with 8 triangles $\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, w \rangle, \langle v_3, v_4, w \rangle, \langle v_4, v_5, w \rangle, \langle v_5, v, w \rangle, \langle v_1, v, w \rangle, \langle v_1, v_6, v \rangle, \langle v_5, v_6, v \rangle$. Because the original triangulation also had 8 triangles and it was minimal, the new triangulation may also be considered. Since (v_1, v_3, v_4, v_5, v) is a 5-cycle in the link of w in D , according to Step 2, a forbidden situation has occurred. So we have reached a contradiction.

If the diagonal of the cycle β' is $\langle v'_2, v'_5 \rangle$, note that (v'_1, v'_2, v'_5, v') is a 4-cycle in $X_{w'}$. Since X is a 5/9-complex, this cycle is not allowed to be full and thus it must have a diagonal. If $v' \sim v'_2$, because the map

f is simplicial and nondegenerate, in D we have $v_2 \sim v_5$, $v \sim v_2$. So we can triangulate D with 8 triangles: $\langle v_1, v_2, v \rangle$, $\langle v_2, v_5, v \rangle$, $\langle v_2, v_5, w \rangle$, $\langle v_2, v_3, w \rangle$, $\langle v_4, v_5, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_1, v_6, v \rangle$, $\langle v_5, v_6, v \rangle$. Note that the original triangulation of the disc D also had 8 triangles: $\langle v_1, v_2, w \rangle$, $\langle v_2, v_3, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_4, v_5, w \rangle$, $\langle v_1, v, w \rangle$, $\langle v_5, v, w \rangle$, $\langle v_1, v_6, v \rangle$, $\langle v_5, v_6, v \rangle$. So the new triangulation may also be considered. But there is a 4-cycle (v_2, v_3, v_4, v_5) in the link of w in D . This implies, by Step 1, a contradiction.

So the link of every neighbor of a vertex of D whose link is 4-large but not 5-large, is 7-large.



Step 3

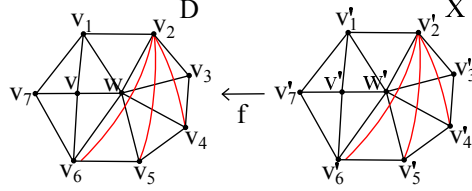
- (d) Step 4: (5/8): every vertex adjacent to a non-5-large (but 4-large) vertex is 8-large.

Assume D is such that $\partial D = \gamma = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ is a cycle formed by two edges $(\langle v_1, v_7 \rangle, \langle v_6, v_7 \rangle)$ in the link of a vertex v and five edges $(\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle, \langle v_3, v_4 \rangle, \langle v_4, v_5 \rangle, \langle v_5, v_6 \rangle)$ in the link of a vertex w . The link of v is 4-large but not 5-large and the link of its neighbor w is 7-large but not 8-large. Let $\beta = (v_1, v_2, v_3, v_4, v_5, v_6, v)$ be the 7-cycle in the link of w . Let $\alpha = (v_1, v, v_6, v_7)$ be the 4-cycle in the link of v . Let γ' be a cycle in X such that (D, f) is a minimal filling diagram for γ' . Since the map f is nondegenerate and simplicial, there exist two vertices v', w' in X such that $f(v) = v'$ and $f(w) = w'$. Moreover, v' has a 4-cycle $\alpha' = (v'_1, w', v'_6, v'_7)$ in its link and w' has a 7-cycle $\beta' = (v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v')$ in its link whereas two of the edges of these cycles coincide (see the figure below). Because X satisfies the 5/9-condition, since v' is 4-large and not 5-large, every its neighbor should be 9-large. Thus, since the 7-cycle in the link of w' is not allowed to be full, it must have a diagonal.

If the diagonal of the cycle β' is $\langle v'_1, v'_6 \rangle$, because the map f is simplicial and nondegenerate, in D we have $v_1 \sim v_6$. Note that then D is not flag. But D can be made flag since, according to Lemma 2.1, we can triangulate D with fewer simplices which implies a contradiction.

Consider further the case when $v'_1 \approx v'_6$. If the diagonal of the cycle β' is $\langle v'_1, v'_3 \rangle$, in D we have $v_1 \sim v_3$. Note that, according to Lemma 2.1 we can triangulate D with 9 triangles $\langle v_1, v_2, v_3 \rangle$, $\langle v_1, v_3, w \rangle$, $\langle v_3, v_4, w \rangle$, $\langle v_4, v_5, w \rangle$, $\langle v_5, v_6, w \rangle$, $\langle v_6, v, w \rangle$, $\langle v_1, v, w \rangle$, $\langle v_1, v_7, v \rangle$, $\langle v_6, v_7, v \rangle$. Note that the new triangulation has the same number of triangles like the original one $(\langle v_1, v_2, w \rangle, \langle v_2, v_3, w \rangle, \langle v_3, v_4, w \rangle, \langle v_4, v_5, w \rangle, \langle v_5, v_6, w \rangle, \langle v_6, v, w \rangle, \langle v_1, v, w \rangle, \langle v_1, v_7, v \rangle, \langle v_6, v_7, v \rangle)$ which was minimal. So it

may also be considered. Since $(v_1, v_3, v_4, v_5, v_6, v)$ is a 6-cycle in the link of w in D , Step 3 implies a contradiction.



Step 4

If the diagonal of the cycle β' is $\langle v'_2, v'_6 \rangle$, note that $(v'_2, v'_3, v'_4, v'_5, v'_6)$ is a 5-cycle in the link of w' in X which is not allowed to be full. So it must have a diagonal. Assume $v'_2 \sim v'_4$. Then we obtain a 4-cycle (v'_2, v'_4, v'_5, v'_6) in $X_{w'}$ which is also not allowed to be full. So it must also have a diagonal, say $\langle v'_2, v'_5 \rangle$. Because the map f is simplicial and non-degenerate, in D we have $v_2 \sim v_4$, $v_2 \sim v_5$, $v_2 \sim v_6$. So we can triangulate D with 9 triangles $\langle v_2, v_3, v_4 \rangle$, $\langle v_2, v_4, v_5 \rangle$, $\langle v_2, v_5, v_6 \rangle$, $\langle v_2, v_6, w \rangle$, $\langle v_1, v_2, w \rangle$, $\langle v_1, v, w \rangle$, $\langle v_6, v, w \rangle$, $\langle v_1, v_7, v \rangle$, $\langle v_6, v_7, v \rangle$. Note that the original triangulation of D also had 9 triangles. So the new triangulation may also be considered. But there is a 5-cycle $(v_2, v_3, v_4, v_5, v_6)$ in the link of w in D which is, according to Step 2, forbidden. So we have reached a contradiction.

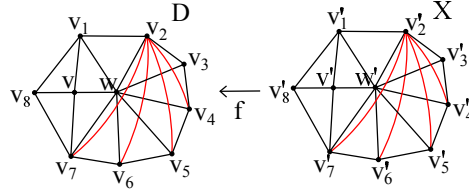
Thus the link of every neighbor of a vertex of D whose link is 4-large but not 5-large, is 8-large.

- (e) Step 5: (5/9): every vertex adjacent to a non-5-large (but 4-large) vertex is 9-large.

Assume D is such that $\partial D = \gamma = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)$ is a cycle formed by two edges in the link of a vertex v ($\langle v_1, v_8 \rangle$, $\langle v_7, v_8 \rangle$) and six edges ($\langle v_1, v_2 \rangle$, $\langle v_2, v_3 \rangle$, $\langle v_3, v_4 \rangle$, $\langle v_4, v_5 \rangle$, $\langle v_5, v_6 \rangle$, $\langle v_6, v_7 \rangle$) in the link of a vertex w . v and w are interior vertices of D such that $v \sim w$. The link of v is 4-large but not 5-large and the link of its neighbor w is 8-large but not 9-large. Let $\beta = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v)$ be the 8-cycle in the link of w . Let $\alpha = (v_1, w, v_7, v_8)$ be the 4-cycle in the link of v . Let γ' be a cycle in X such that (D, f) is a minimal filling diagram for γ' . Since the map f is nondegenerate and simplicial, there exist two vertices v', w' in X such that $f(v) = v'$ and $f(w) = w'$. Moreover, v' has a 4-cycle ($\alpha' = (v'_1, w', v'_7, v'_8)$) in its link and w' has an 8-cycle ($\beta' = (v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v')$) in its link whereas two of the edges of these cycles coincide. Because X satisfies the 5/9-condition, since v' is 4-large and not 5-large, every its neighbor should be 9-large. Thus, since the 8-cycle in the link of w' is not allowed to be full, it must have a diagonal.

If the diagonal of the cycle β' is $\langle v'_1, v'_7 \rangle$, because the map f is simplicial and nondegenerate, in D we have $v_1 \sim v_7$. Note that then D is not flag. But D can be made flag since, according to Lemma 2.1, we can triangulate D with fewer simplices. This implies a contradiction.

Consider further the case when $v'_1 \approx v'_7$. If the diagonal of the cycle β' is $\langle v'_1, v'_3 \rangle$ in D we have $v_1 \sim v_3$. Note that we can triangulate D , according to Lemma 2.1, with 10 triangles $\langle v_1, v_2, v_3 \rangle, \langle v_1, v_3, w \rangle, \langle v_3, v_4, w \rangle, \langle v_4, v_5, w \rangle, \langle v_5, v_6, w \rangle, \langle v_6, v_7, w \rangle, \langle v_7, v, w \rangle, \langle v_1, v, w \rangle, \langle v_7, v_8, v \rangle, \langle v_1, v_8, v \rangle$. Note that the new triangulation has the same number of triangles like the original one ($\langle v_1, v_2, w \rangle, \langle v_2, v_3, w \rangle, \langle v_3, v_4, w \rangle, \langle v_4, v_5, w \rangle, \langle v_5, v_6, w \rangle, \langle v_6, v_7, w \rangle, \langle v_1, v, w \rangle, \langle v_7, v, w \rangle, \langle v_1, v_8, v \rangle, \langle v_7, v_8, v \rangle$) which was minimal. Therefore it may also be considered. Since $(v_1, v_3, v_4, v_5, v_6, v_7, v)$ is a 7-cycle in the link of w in D , we have reached, by Step 4, a contradiction.



Step 5

If the diagonal of the cycle β' is $\langle v'_2, v'_7 \rangle$, note that $(v'_2, v'_3, v'_4, v'_5, v'_6, v'_7)$ is a 6-cycle in $X_{w'}$ which is not allowed to be full. So it must also have a diagonal. If $v'_2 \sim v'_4$, we obtain a 5-cycle $(v'_2, v'_4, v'_5, v'_6, v'_7)$ in $X_{w'}$ which is also not allowed to be full. So it must have a diagonal. If $v'_2 \sim v'_5$, we obtain a 4-cycle (v'_2, v'_5, v'_6, v'_7) in $X_{w'}$ which is not allowed to be full. So it must have a diagonal, say $\langle v'_2, v'_6 \rangle$. Because the map f is simplicial and nondegenerate, in D we have $v_2 \sim v_4$, $v_2 \sim v_5$, $v_2 \sim v_6$, $v_2 \sim v_7$. So we can triangulate D with 10 triangles $\langle v_2, v_3, v_4 \rangle, \langle v_2, v_4, v_5 \rangle, \langle v_2, v_5, v_6 \rangle, \langle v_2, v_6, v_7 \rangle, \langle v_2, v_7, w \rangle, \langle v_1, v_2, w \rangle, \langle v_1, v, w \rangle, \langle v_7, v, w \rangle, \langle v_1, v_8, v \rangle, \langle v_7, v_8, v \rangle$. Note that the original triangulation of the disc D also had 10 triangles. So the new triangulation may also be considered. But there is a 6-cycle $(v_2, v_3, v_4, v_5, v_6, v_7)$ in the link of w in D . Since by Step 3 such cycle is forbidden, we have reached a contradiction.

Hence the link of every neighbor of a vertex of D with 4-large but not 5-large link, is 9-large.

The steps proven above imply that D satisfies the condition (5/9). One can similarly show that D satisfies the condition (6/8) and (7/7). Thus D is a 5/9-complex. \square

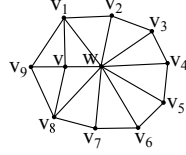
4. THE MAIN RESULT

In this section we prove the paper's main result.

Lemma 4.1 (5/9-diagrams are 8-located). *Let D be a disc satisfying the 5/9-condition. Then D is 8-located.*

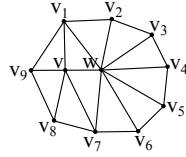
Proof. Note that for any vertex v of D such that $\alpha = (v_1, w, v_8, v_9)$ is a 4-cycle in its link, the 5/9-condition implies that any neighbor w of v has a cycle $\beta = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v)$ of length at least 9 in its link. Hence the length of the

cycle $\gamma = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$ is at least 9. Call such loops γ , loops of type I.



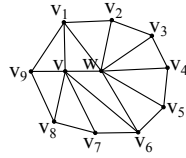
Loop of type I

Note that for any vertex v of D such that $\alpha = (v_1, w, v_7, v_8, v_9)$ is a 5-cycle in its link, the 5/9-condition implies that any neighbor w of v has a cycle $\beta = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v)$ of length at least 8 in its link. Hence the length of the cycle $\gamma = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$ is at least 9. Call such loops γ , loops of type II.



Loop of type II

Note that for any vertex v of D such that $\alpha = (v_1, w, v_6, v_7, v_8, v_9)$ is a 6-cycle in its link, the 5/9-condition implies that any neighbor w of v has a cycle $\beta = (v_1, v_2, v_3, v_4, v_5, v_6, v)$ of length at least 7 in its link. Hence the length of the cycle $\gamma = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$ is at least 9. Call such loops γ , loops of type III.



Loop of type III

Since D is flat, the loops of type I, II and III can not be contained in the link of a vertex. Suppose there exists a loop γ in D of length at most 8 which is not contained in the link of a vertex. We consider further only the situation when $|\gamma| = 8$. The other cases ($|\gamma| \leq 8$) can be treated similarly. Let v, w be two adjacent vertices belonging to the interior of γ such that v is adjacent to some vertices of γ whereas w is adjacent to the other vertices of γ . Denote by α the cycle in the link of v and by β the cycle in the link of w . Since D is flat and the length of γ is 8, we have three possibilities:

- (1) if $|\alpha| = 4$ then $|\beta| = 8$ which yields a contradiction by the condition (5/9) in the definition of 5/9-complexes;
- (2) if $|\alpha| = 5$ then $|\beta| = 7$ which yields a contradiction by the condition (6/8) in the definition of 5/9-complexes;
- (3) if $|\alpha| = 6$ then $|\beta| = 6$ which yields a contradiction by the condition (7/7) in the definition of 5/9-complexes.

Thus no such loops γ exist. So all other loops of D , except for those of type I, II and III, are contained in the link of a vertex. Hence, since the loops of type I, II and III have length at least 9, D is 8-located. \square

Theorem 4.2 (5/9-condition implies 8-location). *Let X be a simply connected 5/9-complex. Then X is 8-located, in particular hyperbolic.*

Proof. Let γ be a cycle of X . Since X is simply connected, there exists a filling diagram (D, f) for γ . We consider this diagram such that it is minimal. According to Lemma 3.1, the disc D is a 5/9-complex. Moreover, Lemma 4.1 implies that D is 8-located. So for any loop γ of length at most 8 of D , there exists a vertex v in D which is adjacent to all vertices of γ . Since the map f is simplicial and nondegenerate, there exists a loop γ' in X such that $|\gamma| = |\gamma'|$ and any edge e of γ is mapped to an edge $e' = f(e)$ of γ' . Also there is a vertex v' of X such that $f(v) = v'$ and v' is adjacent to all vertices of γ' . So any loop γ' of X of length at most 8 is contained in the link of a vertex v' of X . Therefore since X is flag, it follows that X is 8-located. The flagness property is clear due to the 5/9-condition satisfied by the complex X . Since X is simply connected and 8-located, it is, by [Laz15], Theorem 4.3, hyperbolic. \square

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